REPORT ON THE DOCTORAL THESIS
# "LINE ARRANGEMENTS IN ALGEBRAIC TERMS"

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In combinatorial algebraic geometry the study of hyperplane arrangements is one the most classical theories. Historically there have been many original results and the theory of hyperplane arrangements has become a fertile field for research. The problem has ancient roots, if we shrink in the early nineteenth century we have:

**Question 1.** Does an arrangement $\mathcal{L}$ in the real projective plane exist, which is not a pencil of lines, such that there are no double intersection points?

Around 1940, Gallai proved:

**Theorem 1.** *If $\mathcal{L}$ is an arrangement of $d$ lines in the real projective plane which does not contain any double intersection point, then all lines have a common point of intersection, i.e. they form a pencil of $d$ concurrent lines.*

For an arrangement of lines $\mathcal{L}$ in a projective plane we denote by $t_r$ the number of $r$-fold points, i.e., points where exactly $r$ lines from $\mathcal{L}$ meet. In 1941 there was a small improvement of the above theorem:

**Theorem 2** (Melchior). *Let $\mathcal{L} \subset \mathbb{P}^2_{\mathbb{R}}$ be an arrangement of $d \geq 3$ lines with $t_d = 0$, then*

$$t_2 \geq 3 + \sum_{r \geq 3}(r-3)t_r.$$

Then in the complex projective plane, Hirzebruch proved:

**Theorem 3.** *Let $\mathcal{L} \subset \mathbb{P}^2_{\mathbb{C}}$ be an arrangement of $d \geq 6$ lines such that $t_d = t_{d-1} = 0$, then*

$$t_2 + \frac{3}{4}t_3 \geq d + \sum_{r \geq 4}(r-4)t_r.$$

The core of this thesis is the study of a family of Böröczky line arrangements, strictly related to the problem of classification of line arrangements in the real projective plane possessing the maximal possible number of triple intersections points.

The thesis is presented organically and it has a linear and clear structure: after a detailed list of preliminaries about combinatorics of line arrangements, freeness of hyperplane arrangements in the projective space, the author studies parameter spaces of line arrangements for 13, 14, 16, 18 and 24 lines. In particular their parameter spaces are curves of genus $g \geq 2$. Moreover he analyzes the containment problem $I^{(3)} \subset I^2$, where $I$ is the radical ideal of a finite set of points in the projective plane, and $I^{(m)}$ denotes the $m$-th symbolic power, from the combinatorial point of view, that is the following in semplilfied version:

**Theorem 4.** *Let $I \subset \mathbb{K}[x_0, \cdots, x_n]$ be a homogeneous ideal. Then the containment $I^{(m)} \subset I^r$ holds for all $m \geq nr$.*

It's interesting how the author builds the Böröczky arrangements. In detail, he starts from the set of four general points in the projective plane and five lines joining certain pairs of these points. He begins his construction with the four fundamental points:

$$P_1 = (1:0:0) \quad P_2 = (0:1:0) \quad P_3 = (0:0:1) \quad P_4 = (1:1:1).$$

Then he takes the lines joining pairs of these points. Introducing a parameter, he chooses a point on the line joining the points $P_1$ and $P_4$, distinct from all previous points:

$$P_7 = (a:1:1)$$

with the parameter $a \neq 1$ and $a \neq 0$. He builds the lines joining the points $P_3$, $P_7$ and $P_2$, $P_7$, calculating the points of intersection with the other lines. Then he chooses another point on the line joining the points $P_1$ and $P_4$:

$$P_{12} = (b:1:1)$$

with $b \neq a$ and $b \neq 1$. Reiterating the procedure he builds the configuration of $\mathbb{B}_{13}$, analyzing specific degenerate cases. For configurations $\mathbb{B}_{14}$, $\mathbb{B}_{16}$, $\mathbb{B}_{18}$ and $\mathbb{B}_{24}$ the author presents simplified descriptions. It is known that the curves of genus $g \geq 2$ defined over the rationals have only finitely many rational points, by Mordell Conjecture. This conjecture was solved by Faltings, but his proof doesn't give a technique to find them. In fact the author uses an interesting computer algebra programme, named MAGMA, and a MAGMA's command allows to apply the Chabauty's method for finding these rational points. Using the programme, he calculates the rational points of all parameter spaces of Böröczky line arrangements.

The importance of parameter spaces of Böröczky line arrangements is due to the fact that they are used to construct counterexamples to the containment $I^{(3)} \subset I^2$ over the rational numbers (the first counterexample was given by Lampa-Baczynska and Szpond [2], then new counterexamples were given by Harbourne). In fact the core of the Chapter 4 is the following theorem:

**Theorem 5.** *Let us denote $I_3$ the radical ideal of the triple intersection points of Böröczky arrangements of 11 lines. Then the containment $I_3^{(3)} \subset I_3^2$ holds.*

Bokowski and Pokora classified in [1] all oriented matroids of rank 3 with 12 pseudolines and 19 triple points, and they showed that there are only three matroids corresponding to actual arrangements of lines, that are $C_2, C_6$ and $C_7$. The author studies $C_2$ and $C_7$, since he is motivated by the following problem:

**Question 2.** Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two line arrangements having the same number of lines and the same number of the corresponding $t_k$ points, i.e. the same weak combinatorics. Assume that the containment $I_{\mathcal{L}_1}^{(3)} \subset I_{\mathcal{L}_1}^2$ does not hold. Does it follow that the containment $I_{\mathcal{L}_2}^{(3)} \subset I_{\mathcal{L}_2}^2$ is not satisfied?

The question has the negative answer, because the counterexamples are right $C_2$ and $C_7$, in fact the arrangements present the same weak combinatorics and if $I$ is the ideal of triple points, then in $C_2$ arrangements it holds $I_2^{(3)} \subset I_2^2$, but in $C_7$ arrangements it doesn't hold $I_7^{(3)} \subset I_7^2$.

In the final part of the thesis the author studies the freeness of Böröczky line arrangements, in particular he proves the following theorem:

**Theorem 6.** *Let $\mathcal{A} \subset \mathbb{P}^2_{\mathbb{C}}$ be a line arrangement having only double and triple points as the intersections. Suppose that $\mathcal{A}$ is free, then $2 \leq |\mathcal{A}| \leq 9$.*

As consequence it follows the important corollary:

**Corollary 1.** Except the cases $n = 4, 5, 6$, Böröczky arrangements of $n$ lines are not free.

Moreover he focuses attention on the problem of supersolvability numbers, i.e.

**Question 3.** Given a line arrangement $\mathcal{L}$, what is the minimal number of lines to be added to have a supersolvable arrangement?

About Böröczky arrangements of $n$ lines, the author shows that for $n = 6k$ the minimal number of lines to be added is less or equal to $6k^2 - 6k$.

This thesis aims to illustrate these recent developments, as well as to propose new results and ideas which will certainly help to shed new light on this important field of research. The thesis is well set and can be particularly useful to read for those who want to become familiar with the arrangements; furthermore, the numerous original results exhibited are undoubted interesting and certainly deserve to be published in good sector journals. I recommend to accept the thesis.

## References

[1] J. Bokowski, P. Pokora, *On the Sylvester-Gallai and orchad problem for pseudoline arrangements*, Period. Math. Hung., **77(2)** (2018), 164 – 174.

[2] M. Lampa-Baczynska and J. Szpond, *From Pappus Theorem to parameter spaces of some extremal line point configurations and applications*, Geom. Dedicata **188** (2017), 103 – 121.

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